

Reduced Incidence algebras description of cobweb posets and KoDAGs

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Abstract

The notion of reduced incidence algebra of an arbitrary cobweb poset is delivered.

KEY WORDS: cobweb poset, incidence algebra of locally finite poset, order compatible equivalence relation, reduced incidence algebra .

AMS Classification numbers: 06A06, 06A07, 06A11, 11C08, 11B37

Presented at Gian-Carlo Rota Polish Seminar: <http://ii.uwb.edu.pl/akk/sem/semrota.htm>

1 Cobweb posets

The family of the so called cobweb posets Π has been invented by A.K.Kwaśniewski few years ago (for references see: [5, 6]). These structures are such a generalization of the Fibonacci tree growth that allows joint combinatorial interpretation for all of them under the admissibility condition (see [7, 8]).

Let $\{F_n\}_{n \geq 0}$ be a natural numbers valued sequence with $F_0 = 1$ (with $F_0 = 0$ being exceptional as in case of Fibonacci numbers). Any sequence satisfying this property uniquely designates cobweb poset defined as follows.

For $s \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$ let us to define levels of Π :

$$\Phi_s = \{\langle j, s \rangle, \ 1 \leq j \leq F_s\},$$

(in case of $F_0 = 0$ level Φ_0 corresponds to the empty root $\{\emptyset\}$).)

Then

Definition 1. *Corresponding cobweb poset is an infinite partially ordered set $\Pi = (V, \leq)$, where*

$$V = \bigcup_{0 \leq s} \Phi_s$$

are the elements (vertices) of Π and the partial order relation \leq on V for $x = \langle s, t \rangle, y = \langle u, v \rangle$ being elements of cobweb poset Π is defined by formula

$$(x \leq_P y) \iff [(t < v) \vee (t = v \wedge s = u)].$$

Obviously any cobweb poset can be represented, via its Hasse diagram, as infinite directed graf $\Pi = (V, E)$, where set V of its vertices is defined as above and

$$E = \{(\langle j, p \rangle, \langle q, (p+1) \rangle)\} \cup \{(\langle 1, 0 \rangle, \langle 1, 1 \rangle)\},$$

where $1 \leq j \leq F_p$ and $1 \leq q \leq F_{(p+1)}$ stays for set of (directed) edges.

The Kwaśniewski cobweb posets under consideration represented by graphs are examples of oderable directed acyclic graphs (oDAG) which we start to call from now in brief: KoDAGs. These are structures of universal importance for the whole of mathematics - in particular for discrete "mathemagics" [<http://ii.uwb.edu.pl/akk/>] and computer sciences in general (quotation from [7, 8]):

For any given natural numbers valued sequence the graded (layered) cobweb posets' DAGs are equivalently representations of a chain of binary relations. Every relation of the cobweb poset chain is biunivocally represented by the uniquely designated **complete** bipartite digraph-a digraph which is a di-biclique designated by the very given sequence. The cobweb poset is then to be identified with a chain of di-bicliques i.e. by definition - a chain of complete bipartite one direction digraphs. Any chain of relations is therefore obtainable from the cobweb poset chain of complete relations via deleting arcs (arrows) in di-bicliques.

According to the definition above arbitrary cobweb poset $\Pi = (V, \leq)$ is a graded poset (ranked poset) and for $s \in \mathbf{N}_0$:

$$x \in \Phi_s \longrightarrow r(x) = s,$$

where $r : \Pi \rightarrow \mathbf{N}_0$ is a rank function on Π .

Let us then define Kwaśniewski finite cobweb sub-posets as follows

Definition 2. Let $P_n = (V_n, \leq)$, ($n \geq 0$), for $V_n = \bigcup_{0 \leq s \leq n} \Phi_s$ and \leq being the induced partial order relation on Π .

Its easy to see that P_n is ranked poset with rank function r as above. P_n has a unique minimal element $0 = \langle 1, 0 \rangle$ (with $r(0) = 0$). Moreover Π and all P_n s are locally finite, i.e. for any pair $x, y \in \Pi$, the segment $[x, y] = \{z \in \Pi : x \leq z \leq y\}$ is finite.

Let us recall that one defines the incidence algebra of a locally finite partially ordered set P as follows (see [9, 10, 11]):

$$I(P) = I(P, R) = \{f : P \times P \longrightarrow R; \quad f(x, y) = 0 \text{ unless } x \leq y\}.$$

The sum of two such functions f and g and multiplication by scalar are defined as usual. The product $h = f * g$ is defined as follows:

$$h(x, y) = (f * g)(x, y) = \sum_{z \in \mathbf{P}: x \leq z \leq y} f(x, z) \cdot g(z, y).$$

It is immediately verified that this is an associative algebra (with an identity element $\delta(x, y)$, the Kronecker delta), over any associative ring R .

In [4] the incidence algebra of an arbitrary cobweb poset Π (or its subposets P_n) uniquely designated by the natural numbers valued sequence $\{F_n\}_{n \geq 0}$, was considered by the present author. The explicit formulas for some typical elements of incidence algebra $I(\Pi)$ of Π where delivered there.

So for x, y being some arbitrary elements of Π such that $x = \langle s, t \rangle$, $y = \langle u, v \rangle$, ($s, u \in \mathbf{N}$, $t, v \in \mathbf{N}_0$), $1 \leq s \leq F_t$ and $1 \leq u \leq F_v$ one has:

- (1) ζ function of Π being a characteristic function of partial order in Π

$$\zeta(x, y) = \zeta(\langle s, t \rangle, \langle u, v \rangle) = \delta(s, u)\delta(t, v) + \sum_{k=1}^{\infty} \delta(t + k, v), \quad (1)$$

one can also verify, that ζ^k enumerates all multichains of length k ,

- (2) Möbius function of Π being a inverse of ζ

$$\mu(x, y) = \delta(t, v)\delta(s, u) - \delta(t + 1, v) + \sum_{k=2}^{\infty} \delta(t + k, v)(-1)^k \prod_{i=t+1}^{v-1} (F_i - 1), \quad (2)$$

- (3) function $\zeta^2 = \zeta * \zeta$ counting the number of elements in the segment $[x, y]$

$$\zeta^2(x, y) = \text{card}[x, y] = \left(\sum_{i=t+1}^{v-1} F_i \right) + 2, \quad (3)$$

- (4) function η

$$\eta(x, y) = \sum_{k=1}^{\infty} \delta(t + k, v) = \begin{cases} 1 & t < v \\ 0 & w.p.p. \end{cases}, \quad (4)$$

- (5) function $\eta^k(x, y)$, ($k \in \mathbf{N}$) counting the number of chains of length k , (with $(k + 1)$ elements) from x to y

$$\begin{aligned} \eta^k(x, y) &= \sum_{x < z_1 < z_2 < \dots < z_{k-1} < y} 1 \\ &= \sum_{t < i_1 < i_2 < \dots < i_{k-1} < v} F_{i_1} F_{i_2} \dots F_{i_{k-1}}, \end{aligned} \quad (5)$$

(6) function \mathcal{C}

$$\mathcal{C}(\langle s, t \rangle, \langle u, v \rangle) = \delta(t, v)\delta(s, u) - \sum_{k=1}^{\infty} \delta(t + k, v), \quad (6)$$

such that its inverse function $\mathcal{C}^{-1}(x, y)$ counts the number of all chains from x to y

(7) function χ

$$\chi(x, y) = \delta(t + 1, v), \quad (7)$$

(8) function $\chi^k(x, y)$, ($k \in \mathbf{N}$) counting the number of maximal chains of length k , (with $(k + 1)$ elements) from x to y

$$\chi^k(x, y) = \sum_{x \leq z_1 \leq \dots \leq z_{k-1} \leq y} 1 = \delta(t + k, v) F_{t+1} F_{t+2} \dots F_{v-1}. \quad (8)$$

(9) function \mathcal{M}

$$\mathcal{M}(\langle s, t \rangle, \langle u, v \rangle) = \delta(t, v)\delta(s, u) - \delta(t + 1, v), \quad (9)$$

such that its inverse function \mathcal{M}^{-1} counts the number of all maximal chains from x to y .

In this paper the notion of the standard reduced incidence algebra [1, 10, 11] of an arbitrary cobweb poset Π will be delivered. As we shall see, it enables us for example to facilitate the formulas presented above. The results presented below stay true when considering finite subposets P_n defined above.

2 The Standard Reduced Incidence Algebra of an arbitrary cobweb poset

Let $S(\Pi)$ be the set of all segments in Π and let \sim be the equivalence relation $\sim \subseteq S(\Pi) \times S(\Pi)$

Let us recall that \sim is compatible ([1]), i.e. it satisfies the following condition: if f and g belong to the incidence algebra $I(P)$ and $f(x, y) = f(u, v)$ as well as $g(x, y) = g(u, v)$ for all pairs of segments such that $[x, y] \sim [u, v]$, then $(f * g)(x, y) = (f * g)(u, v)$.

The equivalence classes of segments of Π relative to \sim are called types. The set of all functions defined on types (i.e. all functions taking the same value on equivalent segments) forms an associative algebra with identity. One calls it the reduced incidence algebra $R(\Pi, \sim)$ (modulo the the equivalence relation \sim). Let us note that $R(\Pi, \sim)$ is isomorphic to a subalgebra of the $I(\Pi)$, ([1]).

Now let \sim be defined as follows

$$[x, y] \sim [u, v] \iff \text{segments } [x, y], [u, v] \text{ are isomorphic.} \quad (10)$$

One can show that it is order compatible. Then one calls $R(\Pi, \sim) = R(\Pi)$ the standard reduced incidence algebra of Π . Also from the definitions of \sim and partial order on Π one infers that

$$[x, y] \sim [u, v] \iff [r(x) = r(u) \wedge r(y) = r(v)].$$

So let T be the set of types of relation \sim defined above. Then

$$T = \{(k, n) : k, n \in \mathbf{N}_0\}$$

and for $k \leq n$

$$(k, n) = \{[x, y] \in S(\Pi) : r(x) = k \wedge r(y) = n\}, \quad (11)$$

or equivalently

$$(k, n) = \{[x, y] \in S(\Pi) : x \in \Phi_k \wedge y \in \Phi_n\}.$$

Also let $(k, n) = \emptyset$ for $k > n$.

Definition 3. Let $[x, y] \in \alpha_{k, n}$. For $l \in \mathbf{N}_0$ one can define the incidence coefficients in $R(\Pi)$ as follows

$$\left\langle \begin{matrix} k, n \\ l \end{matrix} \right\rangle = |\{z \in [x, y] : [x, z] \in (k, l) \wedge [z, y] \in (l, n)\}|. \quad (12)$$

The the following formula holds.

Proposition 1.

$$\left\langle \begin{matrix} k, n \\ l \end{matrix} \right\rangle = \begin{cases} F_l & k \leq l \leq n \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

Now one can define the product $*$ in $R(\Pi)$ as follows.

Proposition 2. Let $[x, y] \in (k, n)$, $(k \leq n)$ and $f, g \in R(\Pi)$. Then

$$(f * g)(k, n) = \sum_{l \geq 0} F_l f(k, l) g(l, n) \quad (14)$$

with the assumption that $(k, n) = \emptyset$, for $l < k$ or $l > n$.

Proof.

$$\begin{aligned} (f * g)(k, n) &= (f * g)(x, y) = \sum_{x \leq z \leq y} f(x, z) g(z, y) \\ &= \sum_{k \leq l \leq n} \sum_{\{z : [x, z] \in (k, l), [z, y] \in (l, n)\}} f(x, z) g(z, y) \\ &= \sum_{k \leq l \leq n} F_l f(k, l) g(l, n) \\ &= \sum_{l \geq 0} F_l f(k, l) g(l, n), \end{aligned}$$

□

So we have proved

Theorem 1. *Let $F = \{F_n\}_{n \geq 0}$ $z F_0 = 1$, be an arbitrary natural numbers valued sequence with $F_0 = 1$ (with $F_0 = 0$ being exceptional). The the numbers F_n ($n \geq 0$) are the incidence coefficients in the standard reduced algebra $R(\Pi)$ of cobweb poset Π uniquely designated by the sequence $F = \{F_n\}_{n \geq 0}$.*

One can also show the following

Theorem 2. *Let $f \in R(\Pi)$. Then for $x, y \in \Pi$ such that $[x, y] \in (k, n)$ the value $f(x, y)$ depends on $r(x) = k$ and $r(y) = n$ only, i.e.*

$$f(x, y) = f(k, n) = f(r(x), r(y)). \quad (15)$$

From the definition of partial order on Π one can also infer that for $x, y \in \Pi$ satisfying $r(x) = r(y)$ and for $f \in I(\Pi)$, one has

$$f(x, y) = \delta(x, y).$$

It is known that all elements of $I(\Pi)$ mentioned above, i.e. functions: ζ , μ , ζ^2 , ζ^k , η , η^k , \mathcal{C} , \mathcal{C}^{-1} , χ , χ^k , \mathcal{M} , \mathcal{M}^{-1} are the elements of an arbitrary reduced incidence algebra $R(\Pi, \sim)$, (i.e. modulo an arbitrary order compatible equivalence relation \sim on $S(\Pi)$). Then the next results follows immediately from this fact and above theorems.

Corollary 1. *Let $(k, n) \in T$. Then:*

$$\zeta(k, n) = \begin{cases} 1 & k \leq n \\ 0 & k > n \end{cases} ; \quad (16)$$

$$\zeta^2(k, n) = \begin{cases} \left(\sum_{i=k}^{n-1} F_i \right) + 2 & k \leq n \\ 0 & k > n \end{cases} ; \quad (17)$$

$$\eta(k, n) = \delta_{k < n} = \begin{cases} 1 & k < n \\ 0 & k \geq n; \end{cases} \quad (18)$$

$$\eta^2(k, n) = \begin{cases} \sum_{i=k+1}^{n-1} F_i & k \leq n \\ 0 & k > n \end{cases} ; \quad (19)$$

$$\eta^s(k, n) = \begin{cases} \sum_{k < i_1 < \dots < i_{s-1} < n} F_{i_1} F_{i_2} \dots F_{i_{s-1}} & k \leq n \\ 0 & k > n \end{cases} ; \quad (20)$$

$$\mathcal{C}(k, n) = \begin{cases} 1 & k = n \\ -1 & k < n \\ 0 & k > n \end{cases} ; \quad (21)$$

$$\chi(k, n) = \delta(k + 1, n); \quad (22)$$

$$\chi^s(k, n) = \delta(k + s, n) \cdot F_{k+1} F_{k+2} \dots F_{n-1}; \quad (23)$$

$$\mu(k, n) = \begin{cases} (-1)^{n-k} \prod_{i=k+1}^{n-1} (F_i - 1) & k \leq n \\ 0 & k > n \end{cases}. \quad (24)$$

Corollary 2. *The standard reduced incidence algebra $R(\Pi)$ is the maximally reduced incidence algebra $\overline{R}(\Pi)$, i.e. the smallest reduced incidence algebra on Π . Equivalently the equivalence relation defined by (10) is the maximal element in the lattice of all order compatible equivalence relations on $S(\Pi)$.*

Acknowledgements

Discussions with Participants of Gian-Carlo Rota Polish Seminar, http://ii.uwb.edu.pl/akk/sem/sem_rota.htm are highly appreciated.

References

- [1] Doubilet P., Rota G.C., Stanley R.P.: On the foundations of combinatorial theory VI. The idea of generating function, In 6th Berkeley Symp. Math. Stat. Prob. vol. 2 (1972), p. 267-318.
- [2] Krot E.: The first ascent into the Fibonacci Cobweb Poset, Advanced Studies in Contemporary Mathematics 11 (2005), No. 2, p.179-184, ArXiv:math.CO/0411007, cs.DM <http://arxiv.org/abs/math/0411007>
- [3] Krot-Sieniawska E.: On Characteristic Polynomials of the Family of Cobweb Posets, arXiv:0802.2696, cs.DM <http://arxiv.org/abs/0802.2696>, *submitted to: Graphs and Combinatorics*
- [4] Krot-Sieniawska E.: On incidence algebras description of cobweb posets, arXiv:0802.3703, cs.DM <http://arxiv.org/abs/0802.3703>
- [5] Kwaśniewski A.K.: Cobweb posets as noncommutative prefabs, Adv. Stud. Contemp. Math. **14**, 1 (2007), s. 37-47, ArXiv:math/0503286, cs.DM <http://arxiv.org/abs/math/0503286>
- [6] Kwaśniewski A.K.: First observations on Prefab posets' Whitney numbers, Advances in Applied Clifford Algebras Volume 18, Number 1 / February, 2008, p. 57-73. ONLINE FIRST, Springer Link Date, August 10, 2007, arXiv:0802.1696, cs.DM <http://arxiv.org/abs/0802.1696>
- [7] Kwaśniewski A.K.: On cobweb posets and their combinatorially admissible sequences, ArXiv:math.Co/0512578v4 21 Oct 2007, submitted to Graphs and Combinatorics; Japan , cs.DM <http://arxiv.org/abs/math/0512578>

- [8] Kwaśniewski A.K., Dziemiańczuk M.: Cobweb posets - Recent Results, IS-RAMA 2007, December 1-17 2007 Kolkata, INDIA, arXiv:0801.3985, cs.DM <http://arxiv.org/abs/0801.3985>
- [9] Rota G.-C.: On the Foundations of Combinatorial Theory: I. Theory of Möbius Functions, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, vol.2, 1964, p.340-368.
- [10] Spiegel E., O'Donnell Ch.J.: Incidence algebras, Marcel Dekker, Inc. Basel 1997
- [11] Stanley R.P.: Enumerative Combinatorics, Volume I, Wadsworth& Brooks/Cole Advanced Books & Software, Monterey California, 1986.